

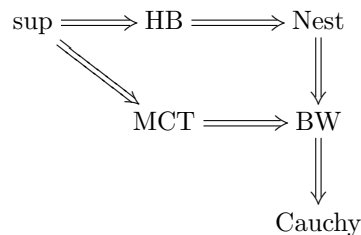
### Assignment 3.

This homework is due *Thursday*, September 18.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 5.

#### 1. EXERCISES

- (1) (1.4.37) Show that every open set in  $\mathbb{R}$  can be represented as a countable union of closed sets. (*Hint*: Use classification of open sets in  $\mathbb{R}$ .)
- (2) (1.4.36) Show that the Borel  $\sigma$ -algebra  $\mathcal{B}$  is the smallest  $\sigma$ -algebra  $\mathcal{A}$  that contains all intervals of the form  $[a, b)$ , where  $a < b$ . (*Hint*: Show that *both*  $\mathcal{B}$  and  $\mathcal{A}$  contain *both* open intervals and intervals of the form  $[a, b)$ .)
- (3) Show that the Heine–Borel theorem is false for:
  - (a) open covers of an open bounded set. (That is, give an example of an open bounded set and its open cover for which the conclusion of the Heine–Borel theorem fails.)
  - (b) open covers of a closed unbounded set.
- (4) In lectures, the following implications were proved or at least sketched (sup = Completeness Axiom, HB = Heine–Borel Theorem, Nest = Nested Set Theorem, BW = Bolzano–Weierstrass Theorem, MCT = Monotone Convergence Theorem, A.P. = Archimedean Principle):



Prove enough implications to make the top five statements equivalent to each other.

- (5) An extended real number  $c$  (that is,  $c \in \mathbb{R}$ , or  $c = \infty$ , or  $c = -\infty$ ) is called a *cluster point* of a sequence  $\{a_n\}$  if a subsequence of  $\{a_n\}$  converges<sup>1</sup> to  $c$ . Show that the set of all real cluster points of a sequence in  $\mathbb{R}$  is a closed set.
- (6) ( $\sim$ 1.5.40) Prove that a sequence in  $\mathbb{R}$  converges to an extended real number  $a \in \mathbb{R}$  if and only if the set of (extended real) cluster points of this sequence is the singleton  $\{a\}$ .

#### 2. EXTRA EXERCISE

- (7) Prove or disprove. For any closed set  $F \subseteq \mathbb{R}$ , there is a sequence in  $\mathbb{R}$  whose set of real cluster points is precisely  $F$ .

<sup>1</sup>A sequence  $\{a_n\}$  converges to  $\infty$  if  $\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n > N, a_n > \varepsilon$ . Similarly for  $-\infty$ .