MA 635, Fall 2014

Assignment 3.

This homework is due *Thursday*, September 18.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 5.

1. Exercises

(1) (1.4.37) Show that every open set in $\mathbb{R}$ can be represented as a countable union of closed sets. (*Hint:* Use classification of open sets in $\mathbb{R}$.)

(2) (1.4.36) Show that the Borel σ-algebra $\mathcal{B}$ is the smallest σ-algebra $\mathcal{A}$ that contains all intervals of the form $[a, b)$, where $a < b$. (*Hint:* Show that *both* $\mathcal{B}$ and $\mathcal{A}$ contain *both* open intervals and intervals of the form $[a, b)$.)

(3) Show that the Heine–Borel theorem is false for:
   (a) open covers of an open bounded set. (That is, give an example of an open bounded set and its open cover for which the conclusion of the Heine–Borel theorem fails.)
   (b) open covers of a closed unbounded set.

(4) In lectures, the following implications were proved or at least sketched (sup = Completeness Axiom, HB = Heine–Borel Theorem, Nest = Nested Set Theorem, BW = Bolzano–Weierstrass Theorem, MCT = Monotone Convergence Theorem, A.P. = Archimedean Principle):

$$\begin{tikzcd}
\text{sup} \arrow[r, Rightarrow] \arrow[dr, Rightarrow] & \text{HB} \arrow[r, Rightarrow] & \text{Nest} \arrow[d, Rightarrow] \\
& \text{MCT} \arrow[r, Rightarrow] & \text{BW} \arrow[d, Rightarrow] \\
& & \text{Cauchy}
\end{tikzcd}$$

Prove enough implications to make the top five statements equivalent to each other.

(5) An extended real number c (that is, $c \in \mathbb{R}$, or $c = \infty$, or $c = -\infty$) is called a *cluster point* of a sequence $\{a_n\}$ if a subsequence of $\{a_n\}$ converges[1] to c. Show that the set of all real cluster points of a sequence in $\mathbb{R}$ is a closed set.

(6) ($\sim$1.5.40) Prove that a sequence in $\mathbb{R}$ converges to an extended real number $a \in \mathbb{R}$ if and only if the set of (extended real) cluster points of this sequence is the singleton $\{a\}$.

2. Extra exercise

(7) Prove or disprove. For any closed set $F \subseteq \mathbb{R}$, there is a sequence in $\mathbb{R}$ whose set of real cluster points is precisely F.

[1] A sequence $\{a_n\}$ converges to ∞ if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}\ \forall n > N, a_n > \varepsilon$. Similarly for $-\infty$.